Handling Material Discontinuities in the Generalized Finite Element Method to Solve Wave Propagation Problems

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Abstract — In this paper the Generalized Finite Element Method (GFEM) with enriched plane wave is used to solve wave propagation problem in an electrically large domain. To treat problem with interface condition two approaches are investigated: Lagrange Multipliers and the Mortar Element Method. Numerical results are compared to known analytical solution.

I. INTRODUCTION

The Generalized Finite Element Method (GFEM) is a combination of the Finite Element Method (FEM) and the Method of Partition of Unity (MPU) [1]. The GFEM has been proved to be suitable to deal with wave propagation problems, where the classical FEM may requires a prohibitive mesh. The application of GFEM to such problems, through the use of special shape functions, allows optimize the number of unknowns. In this paper we use the analytical solutions of the Helmholtz equation in the form of plane waves to enrich the linear polynomial shape functions based on triangular element. Since they carry important information about the solution a small mesh can used. To ensure the continuity of the field between the two media, we also implement and check the efficiency of the Lagrange Multipliers [2] and the Mortar Element Methods [3].

II. FORMULATION

A. Weak form

The model problem is governed by the Helmholtz equation in a domain as shown in Fig. 1. To truncate the domain a type of Robin condition is imposed over $\Gamma_a \cup \Gamma_b$.

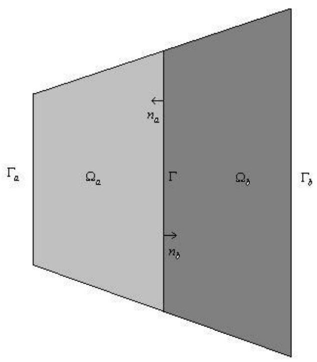

Fig. 1. Subdomains Ω_a and Ω_b with common interface Γ

The weak form in each subdomain is obtained by standard procedure. In Ω_a the weak form is given by

$$\int_{\Omega_a} \left(\nabla v_a \cdot \nabla E_a - k_a{}^2 v_a E_a\right) d\Omega_a + jk_a \int_{\Gamma_a} v_a E_a d\Gamma - \int_{\Gamma} v_a \nabla E_a \cdot n_a d\Gamma = \int_{\Gamma_a} v_a g_a d\Gamma. \quad (1)$$

B. Function space approach in GFEM

To construct the approximation space for GFEM, we combine the FEM functions with the plane wave functions in different directions of the plane. For each element e, the local space of linear combinations of plane wave with directions $\xi_l = \left(\cos\frac{2\pi l}{q}, \sin\frac{2\pi l}{q}\right)$, $l = 1,\ldots,q$ is

$$U_e^{k,q} = \left\{ u = \sum_{i=1}^{n_e} N_i^e u_i \middle| u_i \in W_{local}^{k,q,e} \right\} \quad (2)$$

where

$$W_{local}^{k,q} = span\left\{ w_k^l = \exp\left(jk\left(x\cos\frac{2\pi l}{q} + y\sin\frac{2\pi l}{q} \right)\right)\right\} \quad (3)$$

and q is the number of wave directions. In this case the new enriched shape functions $P_{il}^e = N_i^e w_k^l$ generate the space of approximation functions of GFEM. Thus for each element the electric field is approximated by

$$E^e = \sum_{i=1}^{n_e} \sum_{l=1}^{q} N_i^e w_k^l E_{il}^e = P^e \mathrm{E}^e. \quad (4)$$

III. CONTINUITY BETWEEN SUBDOMAINS

To enforce continuity along the interface Γ, two approaches are used: the Lagrange Multipliers (LM) and the Mortar Element Method (MEM).

A. Lagrange Multipliers

When the LM are used, the contour integral over Γ in (1) is evaluated with

$$\lambda = -\frac{1}{k_a^2}\frac{\partial E_a}{\partial n_a} = \frac{1}{k_b^2}\frac{\partial E_b}{\partial n_b}. \quad (5)$$

and a mixed formulation is obtained.

To approximate the LM, we use the same approach used for the electric field approximation (4), i.e., given an element e with one of its edges belonging to Γ, we write

$$\lambda^e = \sum_{i=1}^{2}\sum_{l=1}^{q} N_i^e w_k^l \lambda_{il}^e \tag{6}$$

where $k = \max(k_a, k_b)$ and λ_{il}^e are the Lagrange multipliers for the node i in direction ξ_l. The matrix form of (6) is given by $\lambda^e = Q^e \lambda_e$, where λ_e is the vector with the Lagrange multipliers λ_{il}^e and $Q_{il}^e = N_i^e w_k^l$. Therefore, we add the continuity condition

$$\int_{\Gamma} Q^e (E_b - E_a) d\Gamma = 0 . \tag{7}$$

Finally, the following symmetric system is obtained:

$$\begin{bmatrix} M_a & . & 0 & . & -C_a^T \\ & . & & . & 0 \\ . \; . & . & . \; . & . & . \\ 0 & . & M_b & . & C_b^T \\ & . & & . & 0 \\ . \; . & . & . \; . & . & . \\ -C_a \;\; 0 & . & C_b \;\; 0 & . & 0 \end{bmatrix} \begin{bmatrix} E_a^{\Gamma} \\ E_a^{o} \\ \cdots \\ E_b^{\Gamma} \\ E_b^{o} \\ \cdots \\ \lambda \end{bmatrix} = \begin{bmatrix} f_a \\ \cdots \\ f_b \\ \cdots \\ 0 \end{bmatrix} \tag{8}$$

where M_b, M_b, f_a and f_b are the conventional stiffness matrices and load vectors of each domain of the GFEM. C_a and C_b are matrices related to integral terms over material interface [2]. E_a^{Γ} and E_b^{Γ} are the vectors of the unknowns amplitudes on Γ, while E_a^o and E_b^o are the ones out of Γ in Ω_a and Ω_b, respectively. The system (8) is ill conditioned and nonpositive definite [2].

B. *Mortar Element Method*

In [3] the MEM is used for enforcing continuity constraints. This method is related to LM through the mortar condition as one can see from (9)

$$E_a^{\Gamma} = C_a^{-1} C_b E_b^{\Gamma} . \tag{9}$$

Using this condition the unknowns over the whole domain, $\Omega_a \cup \Omega_b$, can be linked as follows:

$$\begin{bmatrix} E_a^{\Gamma} \\ E_a^{o} \\ E_b^{\Gamma} \\ E_b^{o} \end{bmatrix} = \begin{bmatrix} 0 & C_a^{-1}C_b & 0 \\ Id & 0 & 0 \\ 0 & Id & 0 \\ 0 & 0 & Id \end{bmatrix} \begin{bmatrix} E_a^{o} \\ E_b^{\Gamma} \\ E_b^{o} \end{bmatrix} \tag{10}$$

which can be expressed as

$$E = \tilde{H}\overline{E} \tag{11}$$

where $\tilde{H}$ is the coupling matrix and Id is the identity matrix.

Applying the coupling matrix on the finite element generalized system $ME = S$, we have

$$\tilde{H}^T M \tilde{H} \overline{E} = \tilde{H}^T S . \tag{12}$$

The resultant system above is sparse and positive definite [3] and, therefore, it can be easily solved by a great range of methods.

IV. RESULTS

Consider the domain $-5 \le x, y \le 5$ where a unit amplitude incident wave in medium a $(k_a = 2\pi)$ travels in $+x$-direction towards medium b $(k_b = 4\pi)$. The interface surface is the $x = 0$ plane.

A. *GFEM with LM*

In this case the domain is discretized with a mesh of 556 elements and 311 nodes. Two plane waves are used to enrich the linear polynomial shape functions. The total number of unknowns is 706. Fig. 2 shows the real and imaginary parts of the electric field along $y = 0$ for the analytical and numerical solutions. As it can be observed there is a good agreement between the solutions. The relative error measured by the energy norm is $1.78E-5$ and $1.12E-5$ for the real end the imaginary parts, respectively.

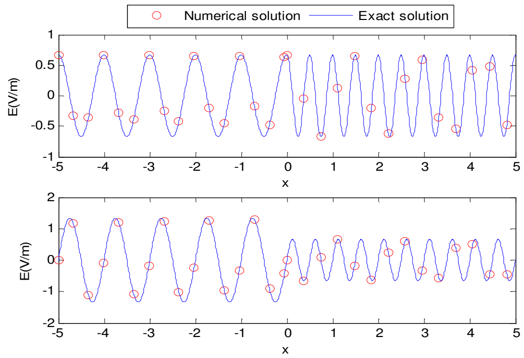

Fig. 2. Plane wave transmitted from a to b using LM

B. *GFEM with MEM*

As expected, in this case we find a resultant system of equations sparse and positive definite. However to get a good result, it is used a mesh of 4783 elements and 2454 nodes. The measured error is 0.022 for the real part and 0.019 for the imaginary part.

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